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NON-SYMMETRIC HALL-LITTLEWOOD POLYNOMIALS

Francois Descouens

Alain Lascoux

À Adriano Garsia, en toute amitié

Abstract

Using the action of the Yang-Baxter elements of the Hecke algebra on polynomials, we define two bases of polynomials in n variables. The Hall-Littlewood polynomials are a subfamily of one of them. For $q = 0$, these bases specialize into the two families of classical Key polynomials (i.e. Demazure characters for type A). We give a scalar product for which the two bases are adjoint of each other.

1 Introduction

We define two linear bases of the ring of polynomials in x_1, \dots, x_n , with coefficients in q .

These polynomials, that we call *q-Key polynomials*, and denote U_v, \widehat{U}_v , $v \in \mathbb{N}^n$, specialize at $q = 0$ into key polynomials K_v, \widehat{K}_v . The polynomials U_v which are symmetrical in x_1, \dots, x_n are precisely the Hall-Littlewood polynomials P_λ , indexed by partitions $\lambda \in \mathfrak{Part}$, the relation between the two indices being $\lambda = [\lambda_1, \dots, \lambda_n] = [v_n, \dots, v_1]$.

Our main tool is the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group, acting on polynomials by deformation of divided differences. This algebra contains two adjoint bases of Yang-Baxter elements (Th. 2.1). The q -Key polynomials are the images of dominant monomials under these Yang-Baxter elements (Def. 3.1). These polynomials are clearly two linear bases of polynomials, since the transition matrix to monomials is uni-triangular.

We show in the last section that $\{U_v\}$ and $\{\widehat{U}_v\}$ are two adjoint bases with respect to a certain scalar product reminiscent of Weyl's scalar product on symmetric functions.

We have intensively used MuPAD (package `MuPAD-Combinat` [11]) and Maple (package `ACE` [10]).

2 The Hecke algebra $\mathcal{H}_n(q)$

Let $\mathcal{H}_n(q)$ be the Hecke algebra of the symmetric group \mathfrak{S}_n , with coefficients the rational functions in a parameter q . It has generators T_1, \dots, T_{n-1} satisfying the braid relations

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i \quad (|j - i| > 1), \end{cases} \quad (1)$$

and the Hecke relations

$$(T_i + 1)(T_i - q) = 0, \quad 1 \leq i \leq n-1 \quad (2)$$

For a permutation σ in \mathfrak{S}_n , we denote by T_σ the element $T_\sigma = T_{i_1} \dots T_{i_p}$ where (i_1, \dots, i_p) is any reduced decomposition of σ . The set $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ is a linear basis of $\mathcal{H}_n(q)$.

2.1 Yang-Baxter bases

Let s_1, \dots, s_{n-1} denote the simple transpositions, $\ell(\sigma)$ denote the length of $\sigma \in \mathfrak{S}_n$, and let ω be the permutation of maximal length.

Given any set of indeterminates $\mathbf{u} = (u_1, \dots, u_n)$, let $\mathcal{H}_n(q)[u_1, \dots, u_n] = \mathcal{H}_n(q) \otimes \mathbb{C}[u_1, \dots, u_n]$.

One defines recursively a *Yang-Baxter basis* $(Y_\sigma^\mathbf{u})_{\sigma \in \mathfrak{S}_n}$, depending on \mathbf{u} , by

$$Y_{\sigma s_i}^\mathbf{u} = Y_\sigma^\mathbf{u} \left(T_i + \frac{1-q}{1-u_{\sigma_{i+1}}/u_{\sigma_i}} \right), \quad \text{when } \ell(\sigma s_i) > \ell(\sigma), \quad (3)$$

starting with $Y_{id}^\mathbf{u} = 1$.

Let φ be the anti-automorphism of $\mathcal{H}_n(q)[u_1, \dots, u_n]$ such that

$$\begin{cases} \varphi(T_\sigma) = T_{\sigma^{-1}}, \\ \varphi(u_i) = u_{n-i+1}. \end{cases}$$

We define a bilinear form $<, >$ on $\mathcal{H}_n(q)[u_1, \dots, u_n]$ by

$$< h_1, h_2 > := \text{coefficient of } T_\omega \text{ in } h_1 \cdot \varphi(h_2). \quad (4)$$

The main result of [6, Th. 5.1] is the following duality property of Yang-Baxter bases.

Theorem 2.1 *For any set of parameters $\mathbf{u} = (u_1, \dots, u_n)$, the basis adjoint to $(Y_\sigma^\mathbf{u})_{\sigma \in \mathfrak{S}_n}$ with respect to $<, >$ is the basis $(\hat{Y}_\sigma^\mathbf{u})_{\sigma \in \mathfrak{S}_n} = (Y_\sigma^{\varphi(\mathbf{u})})_{\sigma \in \mathfrak{S}_n}$. More precisely, one has*

$$\forall \sigma, \nu \in \mathfrak{S}_n, \quad < Y_\sigma^\mathbf{u}, \hat{Y}_\nu^\mathbf{u} > = \delta_{\lambda, \nu\omega}.$$

Let us fix from now on the parameters u to be $\mathbf{u} = (1, q, q^2, \dots, q^{n-1})$. Write \mathcal{H}_n for $\mathcal{H}_n(q)[1, q, \dots, q^{n-1}]$.

In that case, the Yang-Baxter basis $(Y_\sigma)_{\sigma \in \mathfrak{S}_n}$ and its adjoint basis $(\hat{Y}_\sigma)_{\sigma \in \mathfrak{S}_n}$ are defined recursively, starting with $Y_{id} = 1 = \hat{Y}_{id}$, by

$$Y_{\sigma s_i} = Y_\sigma (T_i + 1/[k]_q) \quad \text{and} \quad \hat{Y}_{\sigma s_i} = \hat{Y}_\sigma (T_i + q^{k-1}/[k]_q), \quad \ell(\sigma s_i) > \ell(\sigma), \quad (5)$$

with $k = \sigma_{i+1} - \sigma_i$ and $[k]_q = (1 - q^k)/(1 - q)$.

Notice that the maximal Yang-Baxter elements have another expression [2] :

$$Y_\omega = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \quad \text{and} \quad \hat{Y}_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma\omega)} T_\sigma.$$

Example 2.2 For \mathcal{H}_3 , the transition matrix between $\{Y_\sigma\}_{\sigma \in \mathfrak{S}_3}$ and $\{T_\sigma\}_{\sigma \in \mathfrak{S}_3}$ is

$$\begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \left| \begin{array}{cccccc} 1 & 1 & 1 & \frac{1}{q+1} & \frac{1}{q+1} & 1 \\ \cdot & 1 & \cdot & 1 & \frac{1}{q+1} & 1 \\ \cdot & \cdot & 1 & \frac{1}{q+1} & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array} \right| ,$$

writing ‘ \cdot ’ for 0. Each column represents the expansion of some element Y_σ .

2.2 Action of \mathcal{H}_n on polynomials

Let \mathfrak{Pol} be the ring of polynomials in the variables x_1, \dots, x_n with coefficients the rational functions in q . We write monomials exponentially: $x^v = x_1^{v_1} \dots x_n^{v_n}$, $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. A monomial x^v is *dominant* if $v_1 \geq \dots \geq v_n$.

We extend the natural order on partitions to elements of \mathbb{Z}^n by

$$u \leq v \quad \text{iff} \quad \forall k > 0, \quad \sum_{i=k}^n (v_i - u_i) \geq 0.$$

For any polynomial P in \mathfrak{Pol} , we call *leading term* of P all the monomials (multiplied by their coefficients) which are maximal with respect to this partial order. This order is compatible with the right-to-left lexicographic order, that we shall also use. We also use the classical notation $\mathbf{n}(v) = 0v_1 + 1v_2 + 2v_3 + \dots + (n-1)v_n$.

Let i be an integer such that $1 \leq i \leq n-1$. As an operator on \mathfrak{Pol} , the simple transposition s_i acts by switching x_i and x_{i+1} , and we denote this action by $f \rightarrow f^{s_i}$. The i -th *divided difference* ∂_i and the i -th *isobaric divided difference* π_i , written on the right of the operand, are the following operators :

$$\partial_i : f \mapsto f \partial_i := \frac{f - f^{s_i}}{x_i - x_{i+1}} \quad , \quad \pi_i : f \mapsto f \pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}} .$$

The Hecke algebra \mathcal{H}_n has a faithful representation as an algebra of operators on \mathfrak{Pol} given by the following equivalent formulas [2, 8]

$$\left\{ \begin{array}{llll} T_i & = & \square_i - 1 & = (x_i - qx_{i+1}) \partial_i - 1 = (1 - qx_{i+1}/x_i) \pi_i - 1, \\ Y_{s_i} & = & \square_i & = (x_i - qx_{i+1}) \partial_i = (1 - qx_{i+1}/x_i) \pi_i, \\ \hat{Y}_{s_i} & = & \nabla_i & = \square_i - (1 + q) = \partial_i (x_{i+1} - qx_i). \end{array} \right.$$

The Hecke relations imply

$$\square_i^2 = (1 + q) \square_i \quad , \quad \nabla_i^2 = -(1 + q) \nabla_i \quad \text{and} \quad \square_i \nabla_i = \nabla_i \square_i = 0 .$$

One easily checks that the operators $R_i(a, b)$ and $S_i(a, b)$ defined by

$$R_i(a, b) = \square_i - q \frac{[b - a - 1]_q}{[b - a]_q} \quad \text{and} \quad S_i(a, b) = \nabla_i + q \frac{[b - a - 1]_q}{[b - a]_q}$$

satisfy the Yang-Baxter equation

$$R_i(a, b) R_{i+1}(a, c) R_i(b, c) = R_{i+1}(c, b) R_i(a, c) R_{i+1}(a, b). \quad (6)$$

We have implicitly used these equations in the recursive definition of Yang-Baxter elements (5).

This realization comes from geometry [3], where the maximal Yang-Baxter elements are interpreted as Euler-Poincaré characteristic for the flag variety of $GL_n(\mathbb{C})$. This gives still another expression of the maximal Yang-Baxter elements :

$$Y_\omega = \prod_{1 \leq i < j \leq n} (x_i - qx_j) \partial_\omega \quad , \quad \hat{Y}_\omega = \partial_\omega \prod_{1 \leq i < j \leq n} (x_j - qx_i). \quad (7)$$

Example 2.3 Let $\sigma = (3412) = s_2 s_3 s_1 s_2$. The elements Y_{3412} and \hat{Y}_{3412} can be written

$$\begin{aligned} Y_{3412} &= \square_2 \left(\square_3 - \frac{q}{1+q} \right) \left(\square_1 - \frac{q}{1+q} \right) \left(\square_2 - \frac{q+q^2}{1+q+q^2} \right), \\ \hat{Y}_{3412} &= \nabla_2 \left(\nabla_3 + \frac{q}{1+q} \right) \left(\nabla_1 + \frac{q}{1+q} \right) \left(\nabla_2 + \frac{q+q^2}{1+q+q^2} \right). \end{aligned}$$

We shall now identify the images of dominant monomials under the maximal Yang-Baxter operators with Hall-Littlewood polynomials. Recall that there are two proportional families $\{P_\lambda\}$ and $\{Q_\lambda\}$ of Hall-Littlewood polynomials. Given a partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r] = (0^{m_0}, 1^{m_1}, \dots, n^{m_n})$, with $m_0 = n - r = n - m_1 - \dots - m_n$, then

$$Q_\lambda = \prod_{1 \leq i \leq n} \prod_{j=1}^{m_i} (1 - q^j) P_\lambda.$$

Let moreover $d_\lambda(q) = \prod_{0 \leq i \leq n} \prod_{j=1}^{m_i} [j]_q$. The definition of Hall-Littlewood polynomials with raising operators [7], [9, III.2] can be rewritten, thanks to (7), as follows.

Proposition 2.4 Let λ be a partition of n . Then one has

$$x^\lambda Y_\omega d_\lambda(q)^{-1} = P_\lambda(x_1, \dots, x_n; q) \quad (8)$$

The family of the Hall-Littlewood functions $\{Q_\lambda\}$ indexed by partitions can be extended into a family $\{Q_v : v \in \mathbb{Z}^n\}$, using the following relations due to Littlewood ([7], [9, II.2.Ex. 2])

$$Q_{(\dots, u_i, u_{i+1}, \dots)} = -Q_{(\dots, u_{i+1}-1, u_i+1, \dots)} + q Q_{(\dots, u_{i+1}, u_i, \dots)} + q Q_{(\dots, u_i+1, u_{i+1}-1, \dots)} \quad \text{if } u_i < u_{i+1}, \quad (9)$$

$$Q_{(u_1, \dots, u_n)} = 0 \quad \text{if} \quad u_n < 0 \quad . \quad (10)$$

By iteration of the first relation, one can write any Q_u in terms of Hall-Littlewood functions indexed by decreasing vectors v such that $|v| = |u|$. Consequently, if u such that $|u| = 0$, Q_u must be proportionnal to $Q_{0\dots 0} = 1$, i.e. is a constant that one can note as the specialisation $Q_u(0)$ in $x_1 = 0 = \dots = x_n$.

The final expansion of Q_u , after iterating (9) many times, is not easy to predict. In particular, one needs to know whether $Q_u \neq 0$. For that purpose, we shall isolate a distinguished term in the expansion of Q_u . Given a sum $\sum_{\lambda \in \mathfrak{Part}} c_\lambda(t) Q_\lambda$, call *top term* the image of the leading term $\sum c_\mu(t) Q_\mu$ after restricting each coefficient $c_\mu(t)$ to its term in highest degree in t .

Given $u \in \mathbb{Z}^n$, define recursively $\mathfrak{p}(u) \in \mathfrak{Part} \cup \{-\infty\}$ by

- if $u \not\geq [0, \dots, 0]$ then $\mathfrak{p}(u) = -\infty$
- if $u_2 \geq u_3 \geq \dots \geq u_n > 0$ then $\mathfrak{p}(u)$ is the maximal partition of length $\leq n$, of weight $|u|$ (eventual zero terminal parts are suppressed).
- $\mathfrak{p}(u) = \mathfrak{p}(u \mathfrak{p}([u_2, \dots, u_n]))$

Lemma 2.5 *Let $u \in \mathbb{Z}^n$. Then*

- if $u \not\geq [0, \dots, 0]$ then $Q_u = 0$,
- if $u \geq [0, \dots, 0]$, let $v = \mathfrak{p}(u)$. Then $Q_u \neq 0$ and its leading term is $q^{n(u)-n(v)} Q_v$.

Proof. Given any decomposition $u = u'.u''$, then one can apply (9) to u'' and write Q_u as a linear combination of terms $Q_{u'v}$ with v decreasing, with $|v| = |u''|$. Therefore, if $|u''| = 0$, then the last components of such v are negative, all $Q_{u'v}$ are 0, and $Q_u = 0$.

If $u \geq [0, \dots, 0]$ and u is not a partition, write $u = [\dots, a, b, \dots]$, with a, b the rightmost increase in u . We apply relation (9), assuming the validity of lemma for the three terms in the RHS :

$$Q_{\dots, a, b, \dots} = -Q_{\dots, b-1, a+1, \dots} + qQ_{\dots, b, a, \dots} + qQ_{\dots, a+1, b-1, \dots}$$

Notice that the first two terms have not necessarily an index $\geq [0, \dots, 0]$, but that $[\dots, a+1, b-1, \dots] \geq [0, \dots, 0]$.

In any case, it is clear that $\mathfrak{p}([\dots, b-1, a+1, \dots]) = p_1 \leq v$, $\mathfrak{p}([\dots, b, a, \dots]) = p_2 \leq v$, and $\mathfrak{p}([\dots, a+1, b-1, \dots]) = v$.

Restricted to top terms, the expansion of the RHS in the basis Q_λ becomes

$$-\left((q^{n(u)+a+1-b-n(v)} + \dots) Q_v\right) + q\left((q^{n(u)+a-b-n(v)} + \dots) Q_v\right) + q\left((q^{n(u)-1-n(v)} + \dots) Q_v\right),$$

where one or two of the first two terms may be replaced by 0, depending on the value of p_1 , or p_2 . In final, the top term of the RHS is $q^{n(u)-n(v)} Q_v$, as wanted. Q.E.D

Example 2.6 For $v = [-2, 3, 2]$,

$$Q_{2,3,2} = (q^3 - q^2)Q_3 + (q^5 + q^4 - q^3 - 2q^2 + q)Q_{21} + (q^4 - q^3 - q^2 + q)Q_{111},$$

and the top term is q^4Q_{111} , since $4 = (0(-2) + 1(3) + 2(2)) - (0(1) + 1(1) + 2(1))$ and $[1, 1, 1] > [2, 1]$, $[1, 1, 1] > [3]$. Notice that the coefficient of Q_{21} is of higher degree.

3 q -Key Polynomials

In this section, we show that the images of dominant monomials under the Yang-Baxter elements Y_σ (resp. \hat{Y}_σ), $\sigma \in \mathfrak{S}_n$ constitute two bases of \mathfrak{Pol} , which specialize into the two families of Demazure characters.

We have already identified in the preceding section the images of dominant monomials under Y_ω to Hall-Littlewood polynomial, using the relation between Y_ω and ∂_ω . The other polynomials are new.

3.1 Two bases

The dimension of the linear span of the image of a monomial x^v under all permutations depends upon the stabilizer of v . We meet the same phenomenon when taking the images of a monomial under Yang-Baxter elements.

Let $\lambda = [\lambda_1, \dots, \lambda_n]$ be a decreasing partition (adding eventual parts equal to 0). Denote its orbit under permutations of components by $\mathcal{O}(\lambda)$. Given any v in $\mathcal{O}(\lambda)$, let $\zeta(v)$ be the permutation of maximal length such that $\lambda \zeta(v) = v$ and $\eta(v)$ be the permutation of minimal length such that $\lambda \eta(v) = v$. These two permutations are representative of the same coset of \mathfrak{S}_n modulo the stabilizer of λ .

Definition 3.1 For all v in \mathbb{N}^n , the q -Key polynomials U_v and \hat{U}_v are the following polynomials :

$$U_v(x; q) = \left(\frac{1}{d_\lambda(q)} x^\lambda \right) Y_{\zeta(v)} \quad , \quad \hat{U}_v(x; q) = x^\lambda \hat{Y}_{\eta(v)},$$

where λ is the dominant reordering of v .

In particular, if v is (weakly) increasing, then $\zeta(v) = \omega$ and U_v is a Hall-Littlewood polynomial.

Lemma 3.2 The leading term of U_v and \hat{U}_v is x^v . Consequently, the transition matrix between the U_v (resp. the \hat{U}_v) and the monomials is upper unitriangular with respect to the right-to-left lexicographic order.

Proof. Let k be an integer and u be a weight such that $u_k > u_{k+1}$. Suppose by induction that x^u is the leading term of U_u . Recall the the explicit action of \square_k is (noting only the

two variables x_k, x_{k+1})

$$\begin{aligned} x^{\beta\alpha} \square_k &= x^{\beta\alpha} + (1-t)(x^{\beta-1,\alpha+1} + \dots + x^{\alpha+1,\beta-1}) + x^{\alpha\beta}, \quad \beta > \alpha \\ x^{\beta\beta} \square_k &= (1+t)x^{\beta\beta} \\ x^{\alpha\beta} \square_k &= tx^{\beta\alpha} + (t-1)(x^{\beta-1,\alpha+1} + \dots + x^{\alpha+1,\beta-1}) + tx^{\alpha\beta}, \quad \alpha < \beta. \end{aligned}$$

From these formulas, it is clear that for any constant c , the leading term of $x^u (\square_k + c)$ is $(x^u)^{s_k}$, and, for any v such that $v < u$, all the monomials in $x^v (\square_k + c)$ are strictly less (with respect to the partial order) than $(x^u)^{s_k}$. \square

Example 3.3 For $n = 3$, Figures 1 and 2 show the case of a regular dominant weight x^{210} and Figures 3 and 4 correspond to a case, x^{200} , where the stabilizer is not trivial. In this last case, the polynomials belonging to the family are framed, the extra polynomials denoted A, B do not belong to the basis.

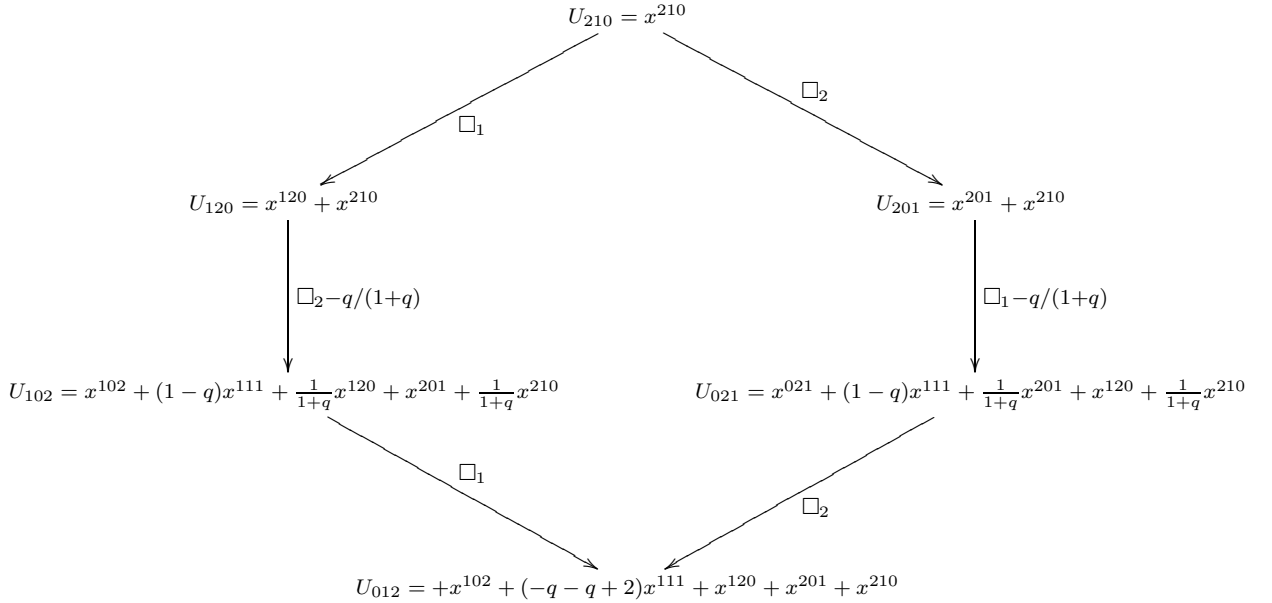


Figure 1: q -Key polynomials generated from x^{210} .

3.2 Specialization at $q = 0$

The specialization at $q = 0$ of the Hecke algebra is called the *0-Hecke algebra*. The elementary Yang-Baxter elements specialize in that case into

$$Y_{s_i} = T_i + 1 = \square_i \rightarrow x_i \partial_i = \pi_i, \quad (11)$$

$$\hat{Y}_{s_i} = T_i = \nabla_i \rightarrow \partial_i x_{i+1} = \hat{\pi}_i. \quad (12)$$

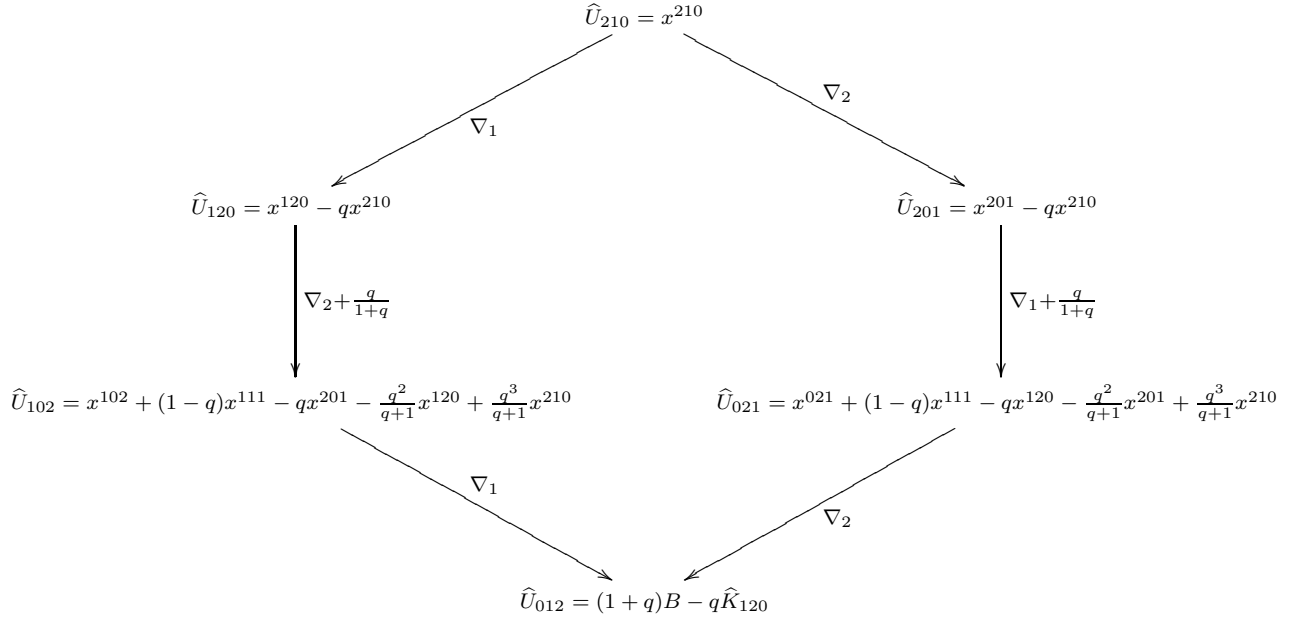


Figure 2: Dual q -Key polynomials generated from x^{210} .

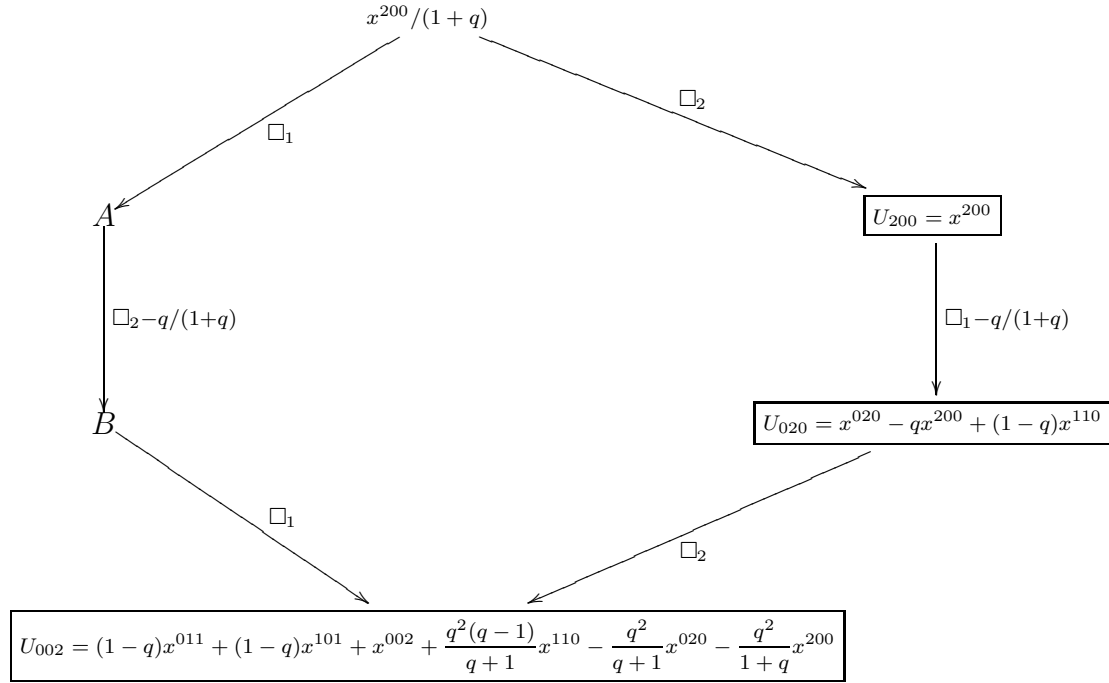


Figure 3: q -Key polynomials generated from $x^{200}/(1+q)$.

Definition 3.4 (Key polynomials) Let $v \in \mathbb{N}^n$. The Key polynomials K_v and \hat{K}_v are

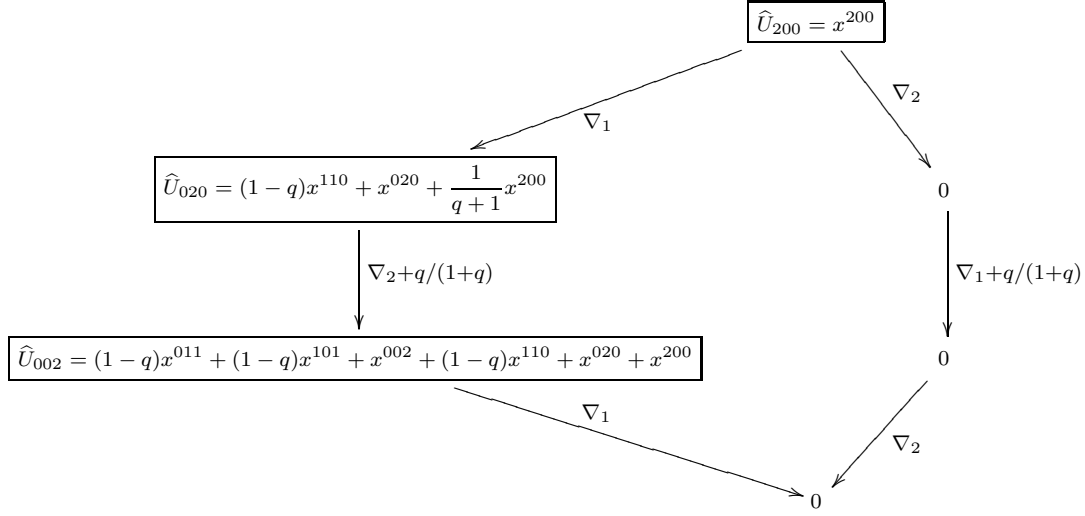


Figure 4: Dual q -Key polynomials generated from x^{200} .

defined recursively, starting with $K_v = x^v = \hat{K}_v$ if x^v dominant, by

$$K_{vs_i} = K_v \pi_i \quad , \quad \hat{K}_{vs_i} = \hat{K}_v \hat{\pi}_i \quad , \quad \text{for } i \text{ such that } v_i > v_{i+1} .$$

In particular, the subfamily (K_v) for v increasing, is the family of Schur functions in x_1, \dots, x_n . Demazure [1] defined Key polynomials (using another terminology) for all the classical groups, and not only the type A_{n-1} which is our case.

Lemma 3.2 specializes into :

Lemma 3.5 *The transition matrix between the U_v and the K_v (resp. from \hat{U}_v to \hat{K}_v) is upper unitriangular with respect to the lexicographic order.*

Example 3.6 *For $n = 3$, the transition matrix between $\{U_v\}$ and $\{K_v\}$ in weight 3 is (reading a column as the expansion of some U_v)*

$$\begin{array}{l}
 300 \\
 210 \\
 201 \\
 120 \\
 111 \\
 102 \\
 030 \\
 021 \\
 012 \\
 003
 \end{array}
 \left|
 \begin{array}{cccccccccc}
 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot & \cdot \\
 \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot \\
 \cdot & \cdot & \cdot & 1 & \cdot & \frac{-q}{(q+1)} & -q & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & -q(q+1) & q^2 \\
 \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
 \end{array}
 \right| ,$$

and the transition matrix between $\{\widehat{U}_v\}$ and $\{\widehat{K}_v\}$ is

$$\begin{array}{l}
300 \\
210 \\
201 \\
120 \\
111 \\
102 \\
030 \\
021 \\
012 \\
003
\end{array}
\begin{vmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & 1 & -q & -q & \cdot & \frac{q^3}{(q+1)} & -q & \frac{q^3}{(q+1)} & -q^3 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & 1 & \cdot & \cdot & -q & \cdot & \frac{-q^2}{(q+1)} & q^2 & -q \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q^2}{(q+1)} & -q & -q & q^2 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & q(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix}.$$

4 Orthogonality properties for the q -Key polynomials

We show in this section that the q -Key polynomials U_v and \widehat{U}_v are two adjoint bases with respect to a certain scalar product.

4.1 A scalar product on \mathfrak{Pol}

For any Laurent series $f = \sum_{i=k}^{\infty} f_i x^i$, we denote by $CT_x(f)$ the coefficient f_0 .

Let

$$\Theta := \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}.$$

Therefore, for any Laurent polynomial $f(x_1, \dots, x_n)$, the expression

$$CT(f \Theta) := CT_{x_n} (CT_{x_{n-1}} (\dots (CT_{x_1} (f \Theta)) \dots))$$

is well defined. Let us use it to define a bilinear form $(,)_q$ on \mathfrak{Pol} by

$$(f, g)_q = CT \left(f g^{\clubsuit} \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j} \right) \quad (13)$$

where \clubsuit is the automorphism defined by $x_i \mapsto 1/x_{n+1-i}$ for $1 \leq i \leq n$.

Since Θ is invariant under \clubsuit , the form $(,)_q$ is symmetrical. Under the specialization $q = 0$, the previous scalar product becomes

$$(f, g) := (f, g)|_{q=0} = CT \left(f g^{\clubsuit} \prod_{1 \leq i < j \leq n} (1 - x_i/x_j) \right). \quad (14)$$

We can also write $(f, g)_q = (f, g\Omega)$ with $\Omega = \prod_{1 \leq i < j \leq n} (1 - qx_i/x_j)^{-1}$.

Notice that, interpreting Schur functions as characters of unitary groups, Weyl defined the scalar product of two symmetric functions f, g in n variables as the constant term of

$$\frac{1}{n!} f g^{\clubsuit} \prod_{i,j:i \neq j} (1 - x_i/x_j).$$

Essentially, Weyl takes the square of the Vandermonde, while we are taking the quotient of the Vandermonde by the q -Vandermonde.

We now examine the compatibility of \square_i and ∇_i with the scalar product.

Lemma 4.1 *For i such that $1 \leq i \leq n-1$, \square_i (resp. ∇_i) is adjoint to \square_{n-i} (resp. ∇_{n-i}) with respect to $(,)_q$.*

Proof. Since π_i (resp. $\widehat{\pi}_i$) is adjoint to π_{n-i} (resp. $\widehat{\pi}_{n-i}$) with respect to $(,)$ (see [5] for more details), we have

$$\begin{aligned} (f \square_i, g)_q &= (f, g \Omega \pi_{n-i} (1 - qx_{n-i+1}/x_{n-i})) \\ &= (f, g \frac{(1 - qx_{n-i+1}/x_{n-i})}{(1 - qx_{n-i+1}/x_{n-i})} \Omega \pi_{n-i} (1 - qx_{n-i+1}/x_{n-i})) \end{aligned}$$

Since the polynomial $\Omega/(1 - qx_{n-i+1}/x_{n-i})$ is symmetrical in the indeterminates x_{n-i} and x_{n-i+1} , it commutes with the action of π_{n-i} . Therefore

$$(f \square_i, g)_q = (f, g (1 - qx_{n-i+1}/x_{n-i}) \pi_{n-i} \Omega) = (f, g \square_{n-i})_q.$$

This proves that \square_i is adjoint to \square_{n-i} , and, equivalently, that ∇_i is adjoint to ∇_{n-i} . Q.E.D

We shall need to characterize whether the scalar product of two monomials vanishes or not. Notice that, by definition,

$$(x^u, x^v) = (x^{u-v\omega}, 1),$$

so that one of the two monomials can be taken equal to 1.

Lemma 4.2 *For any $u \in \mathbb{Z}^n$, then $(x^u, 1)_q \neq 0$ iff $|u| = 0$ and $u \geq [0, \dots, 0]$. In that case, $(x^u, 1)_q = Q_u(0)$.*

Proof. Let us first show that the scalar products $(x^u, 1)_q$ satisfy the same relations (9) as the Hall-Littlewood functions Q_u .

Let k be a positive integer less than n . Write $x_k = y$, $x_{k+1} = z$. Any monomial x^v can be written $x^t y^a z^b$, with x^t of degree 0 in x_k, x_{k+1} . The product

$$x^t (y^a z^b + y^b z^a) (z - qy) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}$$

is equal to

$$(y^a z^b + y^b z^a)(z - qy) \frac{1 - y/z}{1 - qy/z} F_1 = (y^a z^b + y^b z^a)(z - y) F_1$$

with F_1 symmetrical in y, z . The constant term $CT_{x_{k-1}} \dots CT_{x_1}(x^t(y^a z^b + y^b z^a)F_1) = F_2$ is still symmetric in x_k, x_{k+1} . Therefore

$$CT_y \left(CT_z((z - y)F_2) \right)$$

is null, and in final

$$CT \left(x^t(y^a z^b + y^b z^a)(z - qy) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{1 - qx_i/x_j} \right) = 0.$$

This relation can be rewritten

$$(y^a z^{b+1} x^t, 1)_q + (y^{b+1} z^{a+1} x^t, 1)_q - q(y^{b+1} z^a x^t, 1)_q - q(y^{a+1} z^b x^t, 1)_q = 0,$$

which is, indeed, relation (9).

On the other hand, if $u_n < 0$, then there is no term of degree 0 in x_n in $x^u \prod_{1 \leq i < j \leq n} (1 - x_i/x_j)(1 - qx_i/x_j)^{-1}$, and $(x^u, 1) = 0$, so that rule (10) is also satisfied.

In consequence, the function $u \in \mathbb{Z}^n \rightarrow (x^u, 1)$ is determined by the values $(x^\lambda, 1)$, λ partition, as the function $u \in \mathbb{Z}^n \rightarrow Q_u$ is determined by its restriction to partitions. However, for degree reasons, $(x^\lambda, 1) = 0$ if $\lambda \neq 0$. Since $(x^0, 1) = 1$, one has in final that $(x^u, 1) = Q_u(0)$. Q.E.D

Example 4.3 For $u = [1, 0, 3]$ and $v = [0, 1, 3]$,

$$(x^{103}, x^{013})_q = (x^{-2, -1, 3}, 1)_q = Q_{-2, -1, 3}(0) = q^2(1 - q)(1 - q^2).$$

4.2 Duality between $(U_v)_{v \in \mathbb{N}^n}$ and $(\widehat{U}_v)_{v \in \mathbb{N}^n}$

Using that \square_i is adjoint to \square_{n-i} , we are going to prove in this section that U_v and \widehat{U}_v are two adjoint bases of \mathfrak{Pol} with respect to the scalar product $(,)_q$.

We first need some technical lemmas, to allow an induction on the q -Key polynomials, starting with dominant weights.

Lemma 4.4 Let i be an integer such that $1 \leq i \leq n-1$, let f_1, f_2, g_1 be three polynomials and b be a constant such that

$$f_2 = f_1(\square_i + b), \quad (f_1, g_1)_q = 0 \quad \text{and} \quad (f_2, g_1)_q = 1.$$

Then the polynomial $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_1, g_2)_q = 1, \quad (f_2, g_2)_q = 0.$$

Proof. Using that ∇_{n-i} is adjoint to \square_i and that $\square_i \nabla_i = 0$, one has

$$(f_2, g_2)_q = (f_1(\square_i + b), g_1(\nabla_{n-i} - b))_q = (f_1(\square_i + b)(\nabla_i - b), g_1)_q \\ = (f_1(-b(1+q) - b^2), g_1)_q = 0.$$

Similarly, we have

$$(f_1, g_2)_q = (f_1, g_1(\nabla_{n-i} - b))_q \\ = (f_1, g_1(\square_{n-i} - 1 - q - b))_q \\ = (f_1(\square_i + b - 1 - q - 2b), g_1)_q = (f_2, g_1)_q = 1.$$

Q.E.D

Corollary 4.5 *Let i be an integer such that $1 \leq i \leq n-1$, let V be a vector space such that $V = V' \oplus \langle f_1, f_2 \rangle$ with $f_2 = f_1(\square_i + b)$ and V' stable under \square_i , and let g_1 such that*

$$(f_1, g_1)_q = 0 \quad \text{and} \quad (f_2, g_1)_q = 1 \quad \text{and} \quad (v, g_1)_q = 0, \quad \forall v \in V'.$$

Then the element $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_2, g_2)_q = 0 \quad \text{and} \quad (f_1, g_2)_q = 1 \quad \text{and} \quad (v, g_2)_q = 0, \quad \forall v \in V'.$$

Lemma 4.6 *Let u and λ be two dominant weights and v and μ two permutations of u and λ respectively. If $(x^v, x^\lambda) \neq 0$ and $(x^u, x^\mu) \neq 0$ then*

$$u = \lambda, \quad v = \lambda\omega \quad \text{and} \quad \mu = u\omega.$$

Proof. Using lemma 4.2, the condition $(x^v, x^\lambda)_q \neq 0$ and $(x^u, x^\mu)_q \neq 0$ implies two systems of inequalities

$$\left\{ \begin{array}{lcl} v_n & \geq & \lambda_1, \\ v_n + v_{n-1} & \geq & \lambda_1 + \lambda_2, \\ \vdots & \vdots & \vdots \\ v_n + \dots + v_1 & \geq & \lambda_1 + \dots + \lambda_n. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \mu_n & \geq & u_1, \\ \mu_n + \mu_{n-1} & \geq & u_1 + u_2, \\ \vdots & \vdots & \vdots \\ \mu_n + \dots + \mu_1 & \geq & u_1 + \dots + u_n. \end{array} \right.$$

The first inequalities of the systems give $v_n \geq \lambda_1 \geq \mu_n \geq u_1 \geq v_n$. Consequently $u_1 = \lambda_1 = v_n = u_n$. By recursion, using the other inequalities, one gets the lemma. Q.E.D

Corollary 4.7 *Let v be a weight and λ a dominant weight. Then,*

$$(U_v, x^\lambda)_q = \delta_{v, \lambda\omega}$$

Proof. Let u be the decreasing reordering of v and σ the permutation such that $U_v = x^u Y_\sigma$. As the leading term of U_v is x^v and using lemma 4.6, we have that $(x^u Y_\sigma, x^\lambda)_q \neq 0$ implies $(x^v, x^\lambda)_q \neq 0$. By denoting \triangle_σ the adjoint of Y_σ with respect to $(,)_q$, we have $(x^u Y_\sigma, x^\lambda)_q = (x^u, x^\lambda \triangle_\sigma)_q \neq 0$. As the leading term of $x^\lambda \triangle_\sigma$ is $x^{\lambda\sigma'}$, where $\lambda\sigma'$ is a permutation of λ , we obtain that $(x^u, x^{\lambda\sigma'})_q \neq 0$. Using lemma 4.6 we conclude that $v = \lambda\omega$.

Q.E.D

Our main result is the following duality property between U_v and \widehat{U}_v .

Theorem 4.8 *The two sets of polynomials $(U_v)_{v \in \mathbb{N}^n}$ and $(\widehat{U}_v)_{v \in \mathbb{N}^n}$ are two adjoint bases of \mathfrak{Pol} with respect to the scalar product $(,)_q$. More precisely, they satisfy*

$$(U_v, \widehat{U}_{uw})_q = \delta_{v,u} .$$

Proof. Let λ be a dominant weight and V the vector space spanned by the U_v for v in $\mathcal{O}(\lambda)$. The idea of the proof is to build by iteration the elements $(\widehat{U}_v)_{v \in \mathcal{O}(\lambda)}$ starting with $x^\lambda = \widehat{U}_\lambda$. By definition of the q -Key polynomials, it exists a constant b such that $U_{\lambda\omega} = U_{\lambda\omega s_1}(\square_1 + b)$. One can write the decomposition $V = V' \oplus \langle U_{\lambda\omega}, U_{\lambda\omega s_1} \rangle$, with V' stable under the action of \square_1 . Using the previous lemma, we have that $(U_{\lambda\omega}, x^\lambda)_q = (U_{\lambda\omega}, \widehat{U}_\lambda)_q = 1$ and $(U_{\lambda\omega s_1}, x^\lambda)_q = (U_{\lambda\omega s_1}, \widehat{U}_\lambda)_q = 0$. Consequently, by lemma 4.5, the function $x^\lambda(\nabla_{n-1} - b) = \widehat{U}_{\lambda s_1}$ satisfy the duality conditions

$$(U_{\lambda\omega}, \widehat{U}_{\lambda s_1})_q = 0 \quad , \quad (U_{\lambda\omega s_1}, \widehat{U}_{\lambda s_1})_q = 1 \quad \text{and} \quad (v, \widehat{U}_{\lambda s_1})_q = 0 \quad \forall v \in V' .$$

By iteration, this proves that for all u, v , one has $(U_v, \widehat{U}_{uw})_q = \delta_{v,u}$. Q.E.D

This theorem implies that the space of symmetric functions and the linear span of dominant monomials are dual of each other, the Hall-Littlewood functions being the basis dual to dominant monomials.

We finally mention that in the case $q = 0$, one has a reproducing kernel, as stated by the following theorem of [4], which gives another implicit definition of the scalar product $(,)$.

Theorem 4.9 *The two families of polynomials $(K_v)_{v \in \mathbb{N}^n}$ and $(\widehat{K}_v)_{v \in \mathbb{N}^n}$ satisfy the following Cauchy formula*

$$\sum_{u \in \mathbb{N}^n} K_u(x) \widehat{K}_{uw}(y) = \prod_{i+j \leq n+1} \frac{1}{1 - x_i y_j} . \quad (15)$$

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